

Invariant Submanifolds of Darboux-KP Chain and Extension of the Discrete KP Hierarchy

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Abstract

Invariant submanifolds of the so-called Darboux-KP chain [6] are investigated. It is shown that restriction of dynamics on some class of invariant submanifolds yields the extension of the discrete KP hierarchy while the intersections of the latter lead to Lax pairs for a broad class of differential-difference systems with finite number of fields. Some attention is given to investigation of self-similar reductions. It is shown that self-similar ansatzes lead to purely discrete equations with dependence on some number of parameters together with equations governing deformations with respect to these parameters. Some examples are provided. In particular it is shown that well known discrete first Painlevé equation (dPI) corresponds to Volterra lattice hierarchy. It is written down equations which naturally generalize dPI in the sense that they have first Painlevé transcendent in continuous limit.

1 Introduction

In the work [6], was introduced the notion of Darboux-KP (DKP) hierarchy realizing the fundamental concept of Darboux covering adapted for the flows of KP hierarchy. This notion proved to be instrumental for investigation of invariant submanifolds of the KP hierarchy and also for other aims. In fact the DKP hierarchy represents two copies (solutions) of KP hierarchy glued together by Darboux map. Iteration of Darboux map, in both directions yields DKP chain

In the article [15] we constructed two-parameter class of invariant submanifolds of the DKP chain phase space \mathcal{S}_l^n . In particular case $n = 1$, these submanifolds were found in [6]. It was shown there that on \mathcal{S}_0^1 the DKP chain is reduced to discrete KP hierarchy [10], [11].

This work is concerned with investigation of DKP chain phase space invariant submanifolds. In Section 3 we show that restriction of the DKP chain on \mathcal{S}_0^n leads to well-defined Lax equations which can be written down in explicit form. The collection of all the flows of the DKP chain restricted on the class of submanifolds $\{\mathcal{S}_0^n : n \geq 1\}$ naturally form

the extension of the discrete KP hierarchy Restriction of the DKP chain on intersections like $\mathcal{S}_{n,r,l} = \mathcal{S}_0^n \cap \mathcal{S}_{l-1}^{ln-r}$, in turn, gives Lax pairs for differential-difference systems with finite number of fields. These lattices govern Darbox transformations for (restricted) KP hierarchy Lax operators.

To show efficiency of this approach for constructing of integrable lattices, in Section 4, we provide the reader by some examples of integrable lattices, which can be found in the literature and also we construct the class of one-component lattices which naturally includes Bogoyavlenskii ones. We show in this paper that investigation of invariant submanifolds of DKP chain allows to construct Miura transformations between lattices under consideration.

In Section 5 we investigate the solutions of lattice hierarchy invariant with respect to dilatations. It is shown that suitable ansatzes lead to purely discrete equations depending on some collections of parameters together with equations describing deformations of these parameters. Typical example is dPI corresponding to Volterra lattice hierarchy. In particular, we recover the fact that dPI describes Schlesinger transformation of the PIV equation. We investigate class of the discrete equations corresponding to Bogoyavlenskii lattices. All these systems pass singularity confinement test provided some condition on the constants entering these systems. It is shown that this condition is more general than in integrable case. It is shown that all these systems turn into PI in continuous limit.

2 Darboux-KP hierarchy

2.1. KP hierarchy.

First of all let us recall the formalism of the KP hierarchy along the lines proposed in [1], [2], [3]. One considers the space of Laurent series (currents) of the form

$$H^{(0)} = 1, \quad H^{(p)} = z^p + \sum_{l \geq 1} H_l^p z^{-l}.$$

The point of the phase space is defined by semi-infinite matrix $(H_l^p)_{l \geq 1, p \geq 1}$. With each point one associates linear span $\mathcal{H}_+ = \langle 1, H^{(1)}, H^{(2)}, \dots \rangle$ in the space of Laurent series

$$H = \left\{ \sum_{-\infty \ll k < \infty} l_k z^{-k} \right\} = \mathcal{H}_+ \oplus H_-.$$

It is evident that $H_- = \langle z^{-1}, z^{-2}, \dots \rangle$. One considers invariance relation

$$(\partial_p + H^{(p)})\mathcal{H}_+ \subset \mathcal{H}_+, \quad p \geq 1 \tag{1}$$

which one writes in explicit form

$$\partial_p H^{(k)} = H^{(k+p)} - H^{(k)} H^{(p)} + \sum_{s=1}^p H_s^k H^{(p-s)} + \sum_{s=1}^k H_s^p H^{(k-s)}. \tag{2}$$

These equations are called Central System (CS). It is obvious exactness property $\partial_p H^{(k)} = \partial_k H^{(p)}$. Using the latter it is easy to prove the commutativity of SC flows.

The passage to KP hierarchy needs spatialization of some evolution parameter, namely $t_1 = x$. Putting in (2) $p = 1$ one obtains

$$(\partial + h)H^{(k)} = H^{(k+1)} + H_1^k + h_2 H^{(k-1)} + \dots + h_{k+1} \quad (3)$$

where $h = z + h_2 z^{-1} + h_3 z^{-2} + \dots \equiv H^{(1)}$. Using the relation (3) one can easily show that $H^{(k)}$'s are expressed as linear combinations on Faà di Bruno differential polynomials (iterations) which are defined by the following recurrence relations:

$$h^{(k+1)} = (\partial + h)h^{(k)}, \quad h^{(1)} = h, \quad k \geq 1,$$

and under these circumstances one has $\mathcal{H}_+ = \langle 1, h^{(1)}, h^{(2)}, \dots \rangle$. One writes the formula of passage from the basis $\{h^{(k)}\}$ to basis $\{H^{(p)}\}$ as

$$H^{(p)} = h^{(p)} + \sum_{k=0}^{p-2} r_k^p[h] h^{(k)}.$$

On this stage one has passage from the phase space of semi-infinite matrices to KP hierarchy one whose points are defined by infinite collections of (smooth) functions $\{h_k(x), k \geq 2\}$. Using the exactness property one can write

$$\partial_p h = \partial H^{(p)}. \quad (4)$$

Since $H^{(p)} = z^p + \mathcal{O}(z^{-1})$ then $H^{(p)} = \pi_+(z^p)$ where π_+ denotes projection of arbitrary Laurent series into \mathcal{H}_+ .

The relation (4) defines infinite number of evolution equations in the form of local conservation laws. As is known, these equations are entirely equivalent to KP hierarchy while $h(z)$ is interpreted as generating function of Hamiltonian densities. The passage from Laurent series to the algebra of pseudodifferential operators governed by the rule $\phi(h^{(k)}) = \partial^k$ which is extended by linearity on the whole space H . Since negative powers of ∂ enter Lax KP operator, one needs to define negative Faa di Bruno iterations with the help of the relations

$$(\partial + h)h^{(-1)} = 1, \quad (\partial + h)h^{(-2)} = h^{(-1)}$$

and so on. Expressing z as linear combination on Faa di Bruno iterations one defines Lax operator $\mathcal{Q} = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \dots$ as $\mathcal{Q} = \phi(z)$. It is easy from this, to win

$$u_2 = -h_2, \quad u_3 = -h_3, \quad u_4 = -h_4 - h_2^2, \quad u_5 = -h_5 - 3h_2 h_3 + h_2 h_{2x}$$

and so on. It can be seen, that the relation (4) is equivalent to Lax equation $\partial_p \mathcal{Q} = [\mathcal{Q}_+^p, \mathcal{Q}]$.

The relations connecting formal Baker-Akhiezer function $\psi = (1 + \sum_{k \geq 1} w_k z^{-k}) \exp \sum_{p \geq 1} t_p z^p$ with Faà di Bruno iterations and currents are as follows:

$$h^{(k)} = \frac{\partial^k \psi}{\psi}, \quad H^{(p)} = \frac{\partial_p \psi}{\psi}. \quad (5)$$

REMARK 1. The representation of the KP hierarchy in the form of local conservation laws (4) is equivalent to Wilson's theorem [4]. Noncommutative variant of this theorem can be found in [23]. Central System in the form (2) comes back to Cherednik's results (cf. [23]).

2.2. DKP hierarchy.

In the article [6] was defined DKP hierarchy

$$\begin{aligned} \partial_p h &= \partial H^{(p)}, \\ \partial_p a &= a(\tilde{H}^{(p)} - H^{(p)}). \end{aligned} \quad (6)$$

Here laurent series $a = z + \sum_{k \geq 0} a_{k+1} z^{-k}$ differs from h by the presence of zero power of z . "New" currents $\tilde{H}^{(p)}$ are calculated at the point $\tilde{h} = h + a_x/a$.

The equations (6) realize the concept of Darboux covering adapted to dynamical systems of KP hierarchy. Indeed, it is easy to show that having some solution (h, a) of the system (6) one can construct new one with the help of Darboux map $\sigma(h, a) = h + a_x/a$. It was shown in the work [6] that this construction allows enough simply describe many notions using in KP hierarchy theory like Krichever's rational reductions, [7], Darboux and Miura transformations [8], [9], discrete analog of KP hierarchy [10], [11].

One can look at (6) from another viewpoint. Given any pair of KP hierarchy solutions (h, \tilde{h}) or equivalently $(\psi, \tilde{\psi})$ one uniquely defines a . It is easy observe that substitution of $a = z\tilde{\psi}/\psi$ in (6), taking into account (5), turn these equations into identities. In particular the pair (h, h) is suitable for trivial solution $a = z$. From this standpoint the system (6) do not seem to be useful. It becomes really informative after imposing some restrictions compatible with this system. As is known [6], the condition $z^l a \in \mathcal{H}_+$ at $l \geq -1$ defines invariant submanifold \mathcal{S}_l for (6). It is important to observe that in these circumstances KP hierarchy rests to be nonrestricted, but the mapping $h \rightarrow \tilde{h}$ specifies. For instance, on \mathcal{S}_{-1} one has $a = z$ and correspondingly $\tilde{h} = h$. On \mathcal{S}_0 , in turn we have $a = h + a_1$ and

$$\tilde{h} = \frac{a_x + a^2 - a_1 a}{a} = \frac{a^{(2)} - a_1 a^{(1)}}{a}.$$

In terms of wave functions, on \mathcal{S}_0 we have $z\tilde{\psi} = (\partial + a_1)\psi \equiv H\psi$ (elementary Darboux transformation). This transforms is controlled by the function a_1 . It stands to reason that it can not be arbitrary but must satisfy some equations, namely [6]:

$$\partial_p a_1 + \partial \left((-a_1)^{(p)} + \sum_{k=0}^{p-2} r_k^p[h] (-a_1)^{(k)} \right) = 0. \quad (7)$$

Here $(-a_1)^{(k)}$ are corresponding Faà di Bruno iterates

$$(-a_1)^{(0)} = 1, \quad (-a_1)^{(1)} = -a_1, \quad (-a_1)^{(2)} = -a_{1x} + a_1^2$$

and so on. Equation (7) can be exactly linearizable by ansatz $a_1 = -\Phi_x/\Phi$. In a result, one obtains $\partial_p \Phi = \mathcal{Q}_+^p(\Phi)$. τ -function transforms especially simply: $\tilde{\tau} = \Phi\tau$ [12].

Constrained KP hierarchies (Krichever's rational reductions) arises as intersections $\mathcal{S}_l \cap \mathcal{S}_{l+r}$. In this case r -th power of Lax operator turns out to be expressible as a ratio of two purely differential operators

$$\mathcal{Q}^r = P_{l+1}^{-1} Q_{l+r+1} \quad (8)$$

(cf. [13]). In particular, the case $l = -1$ is suitable for reductions to Gelfand-Dickey hierarchies [14].

3 Extension of the discrete KP hierarchy

3.1. DKP chain and its invariant submanifolds.

Applying Darboux iteration to some fixed solution of DKP hierarchy infinitely many times we obtain infinite collection of Laurent series $\{h(i), a(i) : i \in \mathbb{Z}\}$ satisfying the system

$$\begin{aligned} \partial_p h(i) &= \partial H^{(p)}(i), \\ \partial_p a(i) &= a(i)(H^{(p)}(i+1) - H^{(p)}(i)). \end{aligned} \quad (9)$$

REMARK 2. To each copy of KP hierarchy corresponds copy of CS parametrized by some value of i , so $\mathcal{H}_+(i) = \langle 1, H^{(1)}(i), H^{(2)}(i), \dots \rangle$.

As in [15] the system (9) is referred to as DKP chain. In the work [15] was found denumerable class of submanifolds invariant with respect to the flows (9).

THEOREM. *Submanifold \mathcal{S}_l^n defined by the condition*

$$z^{l-n+1} a^{[n]}(i) \in \mathcal{H}_+(i), \quad \forall i \in \mathbb{Z} \quad (10)$$

with $n \in \mathbb{Z}^* \equiv \mathbb{Z}/\{0\}$ is tangent with respect to DKP chain flows.

Here $a^{[k]}(i)$ are “discrete” Faà di Bruno iterates defined by recurrence relations $a^{[k+1]}(i) = a(i)a^{[k]}(i+1)$ with $a^{[0]}(i) \equiv 1$. For $k > 0$,

$$a^{[k]}(i) = a(i)a(i+1)\dots a(i+k-1)$$

and for $k < 0$

$$a^{[k]}(i) = a^{-1}(i-1)a^{-1}(i-2)\dots a^{-1}(i-|k|).$$

PROPOSITION 1. *By virtue of the second equation in (9)*

$$\partial_p a^{[k]}(i) = a^{[k]}(i)(H^{(p)}(i+k) - H^{(p)}(i)). \quad (11)$$

PROOF. For $k > 0$ we have

$$\begin{aligned}\partial_p a^{[k]}(i) &= \partial_p [a(i)a(i+1)\dots a(i+k-1)] = \\ &= \sum_{s=1}^k a(i)\dots a(i+s-1) \{H^{(p)}(i+s) - H^{(p)}(i+s-1)\} a(i+s)\dots a(i+k-1) = \\ &= a^{[k]}(i)(H^{(p)}(i+k) - H^{(p)}(i)).\end{aligned}$$

Analogous calculations are performed for negative k .

COROLLARY. Define, for any integer $k \neq 0$, $h = h(i)$, $\tilde{h} = h(i+k)$, $a = z^{1-k}a^{[k]}(i)$ then, by virtue of (9), the triple (h, \tilde{h}, a) is a solution of DKP hierarchy.

PROPOSITION 2. The following chain of inclusions is valid:

$$\mathcal{S}_l^n \subset \mathcal{S}_{2l+1}^{2n} \subset \dots \subset \mathcal{S}_{kl+k-1}^{kn} \subset \dots \quad (12)$$

PROOF. Let us show that on \mathcal{S}_l^n

$$z^{l-n+1}a^{[n]}(i)\mathcal{H}_+(i+n) \subset \mathcal{H}_+(i). \quad (13)$$

As can be checked the formula (13) follows from the relation

$$z^{l-n+1}a^{[n]}(i)h^{(k)}(i+n) = (\partial + h(i))^k z^{l-n+1}a^{[n]}(i), \quad \forall k \geq 0 \quad (14)$$

together with invariance relation (1). In turn, (14) can be proved by induction. For $k = 0$, (14) is obvious. Let us suppose that this relation is valid for some value of k , then by virtue of (11) we have

$$\begin{aligned}(\partial + h(i))^{k+1} z^{l-n+1}a^{[n]}(i) &= z^{l-n+1}a^{[n]}(i)(h(i+n) - h(i))h^{(k)}(i+n) + \\ &+ z^{l-n+1}a^{[n]}(i)\partial h^{(k)}(i+n) + z^{l-n+1}a^{[n]}(i)h(i)h^{(k)}(i+n) = z^{l-n+1}a^{[n]}(i)h^{(k+1)}(i+n).\end{aligned}$$

Then to show inclusions (12) one needs to use (13) and easily checked formula

$$z^{l-n+1}a^{[n]}(i) \cdot z^{p(l-n+1)}a^{[pn]}(i+n) = z^{(p+1)(l-n+1)}a^{[(p+1)n]}(i). \quad (15)$$

From the condition $z^{l-n+1}a^{[n]}(i) \in \mathcal{H}_+(i)$ and relations (13) and (15) we obtain $z^{2(l-n+1)}a^{[2n]}(i) \in \mathcal{H}_+(i)$ and so on.

DEFINITION. Suppose that the solution of DKP chain is in \mathcal{S}_l^n and there is not invariant submanifold defined by the condition (10) which: 1) is in \mathcal{S}_l^n ; 2) contains given solution then one says that \mathcal{S}_l^n is origin of the chain of inclusions for this solution.

REMARK 3. The reasonings used in proof of Proposition 2 can be found in [15] but this proposition was not exhibited there.

REMARK 4. In the article [15] we suppose that $n \geq 1$ but this is not necessary. From the viewpoint of constructing of integrable lattices consideration of negative values of n

gives not something new since with the help of invertible transformation g_{-1} (see below (37) corresponding to inversion of the discrete parameter: $i \rightarrow -i$, one always can pass to positive n . On the other hand when considering intersections of invariant submanifolds there is a need to use \mathcal{S}_l^n with negative value of n .

The invariance property of \mathcal{S}_l^1 was exhibited in [6]. In the case \mathcal{S}_0^1 DKP chain is reduced to discrete KP hierarchy. In the next section we investigate more general case when $n \geq 1$.

3.2. Extension of the discrete KP hierarchy.

The relation defining \mathcal{S}_0^n looks very simply

$$h(i) = z^{1-n} a^{[n]}(i) - a_1^{[n]}(i) \equiv z^{1-n} a^{[n]}(i) - \sum_{s=1}^n a_1(i+s-1)$$

that is here $h_k(i)$'s are uniquely expressed as polynomials of $a_k(i)$. This means that motion equations on submanifolds \mathcal{S}_0^n can be rewritten only in terms of coordinates $a_k(i)$. More precisely, the equations obtained below (18) define projection of the DKP chain flows from invariant submanifolds on affine hyperplane whose points are parametrized by coordinates $a_k(i)$ (a -surface).

In what follows it will be convenient to define the set of variables $\{q_k^{(n,r)}(i)\}$ as functions of coordinates a -surface with the help of the relation

$$z^r = a^{[r]}(i) + \sum_{k \geq 1} q_k^{(n,r)}(i) z^{k(n-1)} a^{[r-kn]}(i). \quad (16)$$

Since $z^{k(n-1)} a^{[r-kn]}(i)$ is a monic Laurent series of power $r-k$ then the formula (16) uniquely defines $q_k^{(n,r)}(i)$ as polynomials in $a_k(i)$. Remark that this expression has not relation to any invariant submanifold, but it serves for determining the map $\{a_k(i)\} \rightarrow \{q_k^{(n,r)}(i)\}$ for some fixed n and r .

Let us define, for each n , the set of “discrete” currents $\{K_{(n)}^{[pn]}(i) : p \geq 1\}$ with the help of the relation

$$K_{(n)}^{[pn]}(i) = a^{[pn]}(i) + \sum_{k=1}^p q_k^{(n,pn)}(i) z^{k(n-1)} a^{[(p-k)n]}(i).$$

PROPOSITION 3. [15] On \mathcal{S}_0^n

$$z^{(1-n)p} K_{(n)}^{[pn]}(i) = H^{(p)}(i). \quad (17)$$

The relation (17) allows to rewrite second equation in (9) in terms of coordinates $a_k(i)$

$$z^{p(n-1)} \partial_p a(i) = a(i) (K_{(n)}^{[pn]}(i+1) - K_{(n)}^{[pn]}(i)). \quad (18)$$

In these circumstances, by virtue of invariance of \mathcal{S}_0^n the first equation in (9) turn into identity and becomes in a sense unnecessary. Let us observe that the equations (18) describing projections of the flows on different invariant submanifolds are also different what is quite natural. We attach to evolution parameters corresponding to \mathcal{S}_0^n the label: $t_p = t_p^{(n)}$.

As was mentioned above, in the case $n = 1$ the equations (18) are equivalent to discrete KP hierarchy, so it is quite natural to call the whole collection of the flows describing by the system (18) extended discrete KP (edKP) hierarchy. But it is worth to remark that the flows corresponding to different labels do not be obliged to be commutative.

Our next tasks are to rewrite the system (18) in the form of Lax equation; to write down in its explicit form differential-difference equations on variables $q_k^{(n,r)}(i)$ and to list possible reductions of the system (18) for different n . As a result it gives the possibility to cover a broad class of the lattices which admits Lax representation including such classical examples as Volterra and Toda lattices.

3.3. Lax representation for edKP hierarchy.

Let $\{\psi_i : i \in \mathbb{Z}\}$ be the set of wave functions of KP hierarchy corresponding to DKP chain (9). Let us define vector wave function Ψ with coordinates $\Psi_i = z^i \psi_i$. Then

$$\Psi_i = z^i \left(1 + \sum_{k \geq 1} w_k(i) z^{-k}\right) \exp \sum_{p \geq 1} t_p z^p.$$

It is obvious that the relationship between discrete Faà di Bruno iterates and Ψ is defined by the following formulas:

$$a(i) = \frac{\Psi_{i+1}}{\Psi_i}, \quad a^{[r]}(i) = \frac{\Psi_{i+r}}{\Psi_i}. \quad (19)$$

Using the relations (17) and (19), on \mathcal{S}_0^n , one gets

$$\begin{aligned} z^{p(n-1)} H^{(p)}(i) &= z^{p(n-1)} \frac{\partial_p^{(n)} \psi_i}{\psi_i} = z^{p(n-1)} \frac{\partial_p^{(n)} \Psi_i}{\Psi_i} = K_{(n)}^{[pn]}(i) = \\ &= \frac{\Psi_{i+pn}}{\Psi_i} + \sum_{k=1}^p q_k^{(n,pn)}(i) z^{k(n-1)} \frac{\Psi_{i+(p-k)n}}{\Psi_i} \end{aligned}$$

or

$$z^{p(n-1)} \partial_p^{(n)} \Psi = (Q_{(n)}^{pn})_+ \Psi, \quad (20)$$

where

$$(Q_{(n)}^{pn})_+ \equiv \Lambda^{pn} + \sum_{k=1}^p q_k^{(n,pn)}(i) z^{k(n-1)} \Lambda^{(p-k)n},$$

Λ — the shift operator acting by the rule $(\Lambda f)(i) = f(i+1)$.

The equation (20) defines evolutions. To construct Lax pair, one needs to find out eigenvalue problem. From (16) one gets

$$z^r = \frac{\Psi_{i+r}}{\Psi_i} + \sum_{k \geq 1} q_k^{(n,r)}(i) z^{k(n-1)} \frac{\Psi_{i+r-kn}}{\Psi_i}$$

or

$$Q_{(n)}^r \Psi = z^r \Psi, \quad (21)$$

where

$$Q_{(n)}^r \equiv \Lambda^r + \sum_{k \geq 1} q_k^{(n,r)}(i) z^{k(n-1)} \Lambda^{r-kn}.$$

Remark once more that the relation (21) which is simply (16) rewritting in terms of wave vector-function has not relation to any invariant submanifold \mathcal{S}_l^n (as opposed to (20)).

Consistency condition of linear auxiliary linear systems (20) and (21) is the equation

$$z^{p(n-1)} \partial_p^{(n)} Q_{(n)}^r = [(Q_{(n)}^{pn})_+, Q_{(n)}^r]. \quad (22)$$

It can be written in its explicit form

$$\begin{aligned} \partial_p^{(n)} q_k^{(n,r)}(i) &= Q_{k,p}^{(n,r)}(i) = q_{k+p}^{(n,r)}(i+pn) - q_{k+p}^{(n,r)}(i) + \\ &+ \sum_{s=1}^p q_s^{(n,pn)}(i) \cdot q_{k-s+p}^{(n,r)}(i+(p-s)n) - \sum_{s=1}^p q_s^{(n,pn)}(i+r-(k-s+p)n) \cdot q_{k-s+p}^{(n,r)}(i). \end{aligned} \quad (23)$$

REMARK 5. The equation (22) for $r = 1$ is considered [22], [23] and is referred to as gap KP hierarchy. More exactly, for the operator $L = \Lambda + \Lambda^{1-\Gamma} \circ q_0 + \Lambda^{1-2\Gamma} \circ q_1 + \dots$ the equation $\partial_p L = [L_+^{p\Gamma}, L]$ is considered. For this equation, in [23], the problem of integrable discretization of the flow is solved.

In what follows, we exhibit examples of differential-difference systems resulting as different reductions from (23). When constructing such systems it is important to take into account that the functions $q_k^{(n,r)}(i)$ are not independent with respect to each other. The relation (21) says that multiplication wave vector-function by z^r is equivalent to the action on it the operator $Q_{(n)}^r$. Then

$$z^{r_1+r_2} \Psi = Q_{(n)}^{r_1+r_2} \Psi = z^{r_1} Q_{(n)}^{r_2} \Psi = Q_{(n)}^{r_2} Q_{(n)}^{r_1} \Psi = z^{r_2} Q_{(n)}^{r_1} \Psi = Q_{(n)}^{r_1} Q_{(n)}^{r_2} \Psi.$$

From this it follows

$$Q_{(n)}^{r_1+r_2} = Q_{(n)}^{r_1} Q_{(n)}^{r_2} = Q_{(n)}^{r_2} Q_{(n)}^{r_1}$$

or in more explicit form

$$\begin{aligned} q_k^{(n,r_1+r_2)}(i) &= q_k^{(n,r_1)}(i) + \sum_{s=1}^{k-1} q_s^{(n,r_1)}(i) q_{k-s}^{(n,r_2)}(i+r_1-sn) + q_k^{(n,r_2)}(i+r_1) = \\ &= q_k^{(n,r_2)}(i) + \sum_{s=1}^{k-1} q_s^{(n,r_2)}(i) q_{k-s}^{(n,r_1)}(i+r_2-sn) + q_k^{(n,r_1)}(i+r_2). \end{aligned} \quad (24)$$

Directly from (23) one sees that equations corresponding to some fixed values of n and r admit reduction with the help of the restriction

$$q_k^{(n,r)}(i) \equiv 0, \quad k > l, \quad l \geq 1. \quad (25)$$

From geometrical viewpoint these reductions, as was shown in [15], is relevant to intersections $\mathcal{S}_{n,r,l} \equiv \mathcal{S}_0^n \cap \mathcal{S}_{l-1}^{ln-r}$.

4 Integrable lattices

The aim of this section is to exhibit some examples of differential-difference systems which can be derived from (23) and (24). By its construction, they possess Lax pair representation (22) with some reduced operator $Q_{(n)}^r$ and have direct relation to KP hierarchy in the sense that they relate in some way the sequences of (restricted) KP Lax operators.

4.1. Examples.

First of all it should be notice recent work [19] in which a broad class of lattices having Lax pair was exhibited. But it can be shown that the most part of these examples can be extracted from (22).

On $\mathcal{S}_{n,r,l}$ we have $Q_{(n)}^r = \Lambda^r + \sum_{k=1}^l z^{k(n-1)} q_k^{(n,r)} \Lambda^{r-kn}$. Denote $\lambda = z^{1-n}$ and $H(\lambda) = \lambda^l Q_{(n)}^r$. Then

$$H(\lambda) = \lambda^l \Lambda^r + \sum_{k=1}^l \lambda^{l-k} q_k^{(n,r)} \Lambda^{r-kn} = \sum_{k=1}^l \lambda^k H_k, \quad (26)$$

$$\frac{\lambda^p}{\lambda^{Nl}} H^N(\lambda) = \lambda^p \Lambda^{pn} + \sum_{k=1}^{\infty} \lambda^{p-k} q_k^{(n,r)} \Lambda^{(p-k)n}.$$

Observe that Lax equation (22), provided that $pn = Nr$ where N is some ineger, is rewritten in the following form:

$$\partial_p^{(n)} H(\lambda) = \left[\left(\lambda^{p-Nl} H^N(\lambda) \right)_{\infty}, H(\lambda) \right] = - \left[\left(\lambda^{p-Nl} H^N(\lambda) \right)_0, H(\lambda) \right]. \quad (27)$$

Here the subscripts ∞ and 0 denote projections of Laurent series (with matrix coefficients) on nonnegative and negative parts, respectively. The equation (27) is basis for constructing integrable lattices, Darboux transformations and soliton solutions in the work [19]. Unfortunately in this work, there is no indication how to construct matrices H_k , while most part of examples presented in this work are suitable for (26).

The possibility to write down the (23) as the system with finite number of fields in closed form, is that thanks to condition $pn = Nr$, the quantities $q_k^{(n,pn)}$ are polynomially expressed with the help of (24) upon $\rho_k \equiv q_k^{(n,r)}$.

Let us exhibit some lattices which can be found in the literature¹.

EXAMPLE 1. $n \geq 2$, $p = 1$, $r = 1$, $l = 1$, [20], [21]

$$\rho'_i = \rho_i \left(\sum_{s=1}^{n-1} \rho_{i+s} - \sum_{s=1}^{n-1} \rho_{i-s} \right), \quad \rho_1(i) \equiv \rho_i, \quad (28)$$

EXAMPLE 2. $n = p + 1$, $p \geq 1$, $r = p$, $l = 1$, [21]

$$\rho'_i = \rho_i \left(\prod_{s=1}^{n-1} \rho_{i+s} - \prod_{s=1}^{n-1} \rho_{i-s} \right),$$

¹Denote $' = \partial/\partial t_p^{(n)}$ with some corresponding values of p and n

EXAMPLE 3. $n = 1, p = 1, r = 1, l \geq 2$, [22]

$$\begin{aligned}\rho'_1(i) &= \rho_2(i+1) - \rho_2(i), \\ \rho'_k(i) &= \rho_{k+1}(i+1) - \rho_{k+1}(i) + \rho_k(i) (\rho_1(i) - \rho_1(i-k+1)), \quad k = 2, \dots, l-1, \\ \rho'_l(i) &= \rho_l(i) (\rho_1(i) - \rho_1(i-l+1)),\end{aligned}\tag{29}$$

EXAMPLE 4. $n = 1, p = r, r \geq 1, l = p+1$, [24]

$$\begin{aligned}\rho'_1(i) &= \rho_{p+1}(i+p) - \rho_{p+1}(i), \\ \rho'_k(i) &= \rho_{k-1}(i) \rho_{p+1}(i+p-k+1) - \rho_{k-1}(i-1) \rho_{p+1}(i), \quad k = 2, \dots, p+1,\end{aligned}$$

EXAMPLE 5. $n \geq 1, p = 1, r = -1, l = 2$

$$\begin{aligned}\rho'_1(i) &= \rho_2(i+n) - \rho_2(i) + \rho_1(i) \left(\sum_{s=1}^n \rho_1(i-s) - \sum_{s=1}^n \rho_1(i+s) \right), \\ \rho'_2(i) &= \rho_2(i) \left(\sum_{s=1}^n \rho_1(i-s-n) - \sum_{s=1}^n \rho_1(i+s) \right).\end{aligned}\tag{30}$$

The system (28) is known as Bogoyavlenskii lattice (cf. [20]). In particular case $n = 2$, this is Volterra lattice. The system (29) is known in the literature as generalized Toda lattice or Kupershmidt lattice. In particular case $n = 1$, we have ordinary Toda lattice. The system (30), for $n = 1$, was considered in [25]. All these examples and many others can be found in [19].

EXAMPLE 6. $n = 2, p = 1, r = 3, l = 2$ [26]

$$\begin{aligned}(\rho_1(i-1) + \rho_1(i) + \rho_1(i+1))' &= \\ &= (\rho_1(i-1) + \rho_1(i) + \rho_1(i+1)) (\rho_1(i-1) - \rho_1(i+1)) + \rho_2(i+1) - \rho_2(i-1), \\ \rho'_2(i) &= \rho_2(i) (\rho_1(i+1) - \rho_1(i-1)).\end{aligned}\tag{31}$$

$\rho_1(i) \equiv q_1^{(2,1)}(i), \rho_2(i) \equiv q_2^{(2,3)}(i).$

EXAMPLE 7. $n = 1, p = 1, r \geq 2, l = r$

$$\begin{aligned}(\rho_1(i) + \dots + \rho_1(i+r-1))' &= \\ &= (\rho_1(i) + \dots + \rho_1(i+r-1)) (\rho_1(i) - \rho_1(i+r-1)) + \rho_2(i+1) - \rho_2(i), \\ \rho'_k(i) &= \rho_k(i) (\rho_1(i) - \rho_1(i+r-k)) + \rho_{k+1}(i+1) - \rho_{k+1}(i), \quad k = 2, \dots, r-2, \\ \rho'_{r-1}(i) &= \rho_{r-1}(i) (\rho_1(i) - \rho_1(i+1)) + \mu_{i+1} - \mu_i.\end{aligned}\tag{32}$$

Here $\rho_1(i) \equiv q_1^{(1,1)}(i), \rho_k(i) \equiv q_k^{(1,r)}(i); \mu_i = q_r^{(1,r)}(i)$ are constants by virtue of motion equations. The system (32) describes Darboux transformation of purely differential operator of r -th order [18]. Indeed, due to [16] one can write

$$\mathcal{Q}_i^l = H_{i+(l-1)n} \dots H_{i+n} H_i + q_1^{(n,ln)}(i) H_{i+(l-2)n} \dots H_{i+n} H_i + \dots$$

$$+q_l^{(n,ln)}(i) + q_{l+1}^{(n,ln)}(i)H_{i-n}^{-1} + q_{l+2}^{(n,ln)}(i)H_{i-2n}^{-1}H_{i-n}^{-1} + \dots,$$

where $H_i \equiv \partial^{(n)} - q_1^{(n,n)}(i)$ is the operator defining elementary Darboux transformation: $z\psi_{i+n} = H_i\psi_i$. Then on $\mathcal{S}_{1,r,r}$ we have

$$\mathcal{Q}_i^r = H_{i+r-1} \dots H_{i+1} H_i + \rho_1^{(r)}(i) H_{i+r-2} \dots H_{i+1} H_i + \rho_2(i) H_{i+r-3} \dots H_{i+1} H_i + \dots + \mu_i,$$

where $\rho_1^{(r)}(i) \equiv \rho_1(i) + \dots + \rho_1(i+r-1)$. As is known, the condition $\mathcal{Q}_i^r = (\mathcal{Q}_i^r)_+$ defines reduction of KP hierarchy to Gelfand-Dickey one.

REMARK 6. The systems (31) and (32), in contrast with above-mentioned examples do not admit the representation (27), since in this case pn/r 's are ratios.

The rest of this section will be concerned with constructing of some class of one-component lattices which naturally contains Bogoyavlenskii lattices. For simplicity, let us consider the system which describes the evolution on $\mathcal{S}_{4,2,1}$. For $\rho = q_1^{(4,2)}$ we have

$$\partial^{(4)}\rho_i = \rho_i(\rho_{i+2} - \rho_{i-2}).$$

This is Volterra lattice with double spacing. From (24) we have $\rho_i = \sigma_i + \sigma_{i+1}$, where $\sigma \equiv q_1^{(4,1)}$. Moreover, on $\mathcal{S}_{4,2,1}$, we have

$$q_2^{(4,2)}(i) = 0 = q_2^{(4,1)}(i+1) + q_2^{(4,1)}(i) + q_1^{(4,1)}(i-3)q_1^{(4,1)}(i)$$

or

$$q_2^{(4,1)}(i+1) + q_2^{(4,1)}(i) = -\sigma_{i-3}\sigma_i.$$

Then

$$q_2^{(4,1)}(i+4) - q_2^{(4,1)}(i) = \sigma_{i-3}\sigma_i - \sigma_{i-2}\sigma_{i+1} + \sigma_{i-1}\sigma_{i+2} - \sigma_i\sigma_{i+3}. \quad (33)$$

It follows from (23) that

$$\partial^{(4)}\sigma_i = \partial^{(4)}q_1^{(4,1)}(i) = q_2^{(4,1)}(i+4) - q_2^{(4,1)}(i) + q_1^{(4,1)}(i)(q_1^{(4,4)}(i) - q_1^{(4,4)}(i-3)).$$

Taking into account $q_1^{(4,4)}(i) = \sigma_i + \sigma_{i+1} + \sigma_{i+2} + \sigma_{i+3}$ and the relation (33), as a result, we gets

$$\sigma'_i = \sigma_i(\sigma_{i+2} + \sigma_{i+1} - \sigma_{i-1} - \sigma_{i-2}) + \sigma_{i+2}\sigma_{i-1} - \sigma_{i-2}\sigma_{i+1}.$$

Now let us generalize this example. On $\mathcal{S}_{mr,r,1}$, where $m \geq 2$ $r \geq 1$, we have

$$\partial^{(mr)}\rho_i = \rho_i \left(\sum_{s=1}^{m-1} \rho_{i+sr} - \sum_{s=1}^{m-1} \rho_{i-sr} \right).$$

Here $\rho = q_1^{(mr,r)}$. Let $\sigma = q_1^{(mr,1)}$, then $\rho_i = \sigma_i + \dots + \sigma_{i+r-1}$. Let us, omitting technical details, to write down the equations on the field σ

$$\sigma'_i = \sigma_i \left(\sum_{s=1}^{(m-1)r} \sigma_{i+s} - \sum_{s=1}^{(m-1)r} \sigma_{i-s} \right) +$$

$$+ \sum_{s=1}^{m-1} \sigma_{i+sr} \left(\sum_{s_1=1}^{r-1} \sigma_{i-s_1+(s-m+1)} \right) - \sum_{s=1}^{m-1} \sigma_{i-sr} \left(\sum_{s_1=1}^{r-1} \sigma_{i+s_1-(s-m+1)} \right).$$

4.2. The relationship of constrained KP hierarchies with integrable lattices.

As is known, integrable lattices are treated as discrete symmetries (Darboux transformations) for corresponding integrable (differential) hierarchies.

PROPOSITION 4. *Let the solution of the DKP chain is in $\mathcal{S}_{n,r,l}$. Denote $m = ln - r$,*

$$\psi = \psi_i, \quad \tilde{\psi} = \psi_{i+nm}, \quad a = z \frac{\tilde{\psi}}{\psi} = z \frac{\psi_{i+nm}}{\psi_i} = z^{1-nm} a^{[nm]}(i).$$

1) *If $m \geq 0$, then the triple (h, \tilde{h}, a) is a solution of DKP hierarchy such that*

$$z^\ell a \in \mathcal{H}_+, \quad z^{\ell+r} a \in \mathcal{H}_+, \quad (34)$$

where $\ell = m - 1 \geq -1$.

2) *If $m < 0$, then the triple (h, \tilde{h}, a) is a solution of DKP hierarchy satisfyng*

$$z^{-\ell} a^{-1} \in \tilde{\mathcal{H}}_+, \quad z^{\ell+r} a \in \mathcal{H}_+. \quad (35)$$

PROOF. We prove the first part of proposition. The second one can be proved by analogy. By virtue of Proposition 1 and its corollary (h, \tilde{h}, a) is indeed a solution of DKP hierarchy. In the circumstances of this proposition we have

$$z^{1-n} a^{[n]}(i) \in \mathcal{H}_+(i), \quad z^{l(1-n)+r} a^{[ln-r]} \in \mathcal{H}_+(i). \quad (36)$$

Using the reasonings taking into account when proving Proposition 2, as consequence of (36), we obtain

$$z^{(1-n)m} a^{[nm]}(i) \in \mathcal{H}_+(i), \quad z^{(l-m)n} a^{[mn]} \in \mathcal{H}_+(i).$$

These relations, in turn, can be rewritten in the form (34).

As was mentioned above, the relations (34) determine Krichever's rational reductions of KP hierarchy including (for $\ell = -1$) Gelfand-Dickey ones. In this situation r -th power of Lax operator is expressed in the form of ratio of two differential operators (8). Using standard reasonings, one can derive that from (35) it follows, that

$$\mathcal{Q}^r = P_{|\ell+1|} \mathcal{Q}_{\ell+r+1}.$$

It is worthwhile to note some works in which the relationship between integrable lattices and constrained KP hierarchies is mentioned. The article [30] is conserved with discrete symmetries for the so-called multi-boson hierarchies with Lax operator of the form

$$\mathcal{Q} = \partial + \sum_{k=1}^n R_k (\partial - S_k)^{-1} \dots (\partial - S_2)^{-1} (\partial - S_1)^{-1}.$$

These symmetries is given in its explicit form as shifts on generalized Toda lattices (29). In [17] it was constructed the modified version of Krichever's rational reductions. The approach in [17] essentially uses some class of one-component lattices ($\mathcal{S}_{n,r,1}$, $r = 1, \dots, n-1$). In [17] also the discrete symmetries are constructed. The results of these two works as can be shown are compatible between each other and are in agreement with the Proposition 4.

It should be noted here that the relationship between discrete integrable systems and restricted KP hierarchies is considerably used in matrix models (see, for example [31], [32]).

4.2. Lattice Miura transformations.

Given any solution of DKP chain define, for some $k \in \mathbb{Z}^*$, the following transformation:

$$g_k : \begin{cases} a(i) \rightarrow z^{1-k} a^{[k]}(ki), \\ h(i) \rightarrow h(ki). \end{cases} \quad (37)$$

LEMMA. *The relation*

$$\bar{a}^{[r]}(i) = z^{r(1-k)} a^{[rk]}(ki), \quad \forall r, k \in \mathbb{Z}^*, \quad (38)$$

is valid where $\bar{a}(i) \equiv g_k(a(i))$.

PROOF. For $r > 0$, we have

$$\begin{aligned} \bar{a}^{[r]}(i) &\equiv \bar{a}(i) \bar{a}(i+1) \dots \bar{a}(i+r-1) = z^{r(1-k)} a^{[k]}(ki) a^{[k]}(ki+k) \dots a^{[k]}(ki+(r-1)k) = \\ &= z^{r(1-k)} a^{[rk]}(ki), \end{aligned}$$

while for $r < 0$,

$$\begin{aligned} \bar{a}^{[r]}(i) &\equiv \bar{a}^{-1}(i-1) \bar{a}^{-1}(i-2) \dots \bar{a}^{-1}(i-|r|) = \\ &= z^{|r|(k-1)} a^{[-k]}(ki) a^{[-k]}(ki-k) \dots a^{[-k]}(ki-(|r|-1)k) = z^{r(1-k)} a^{[rk]}(ki). \end{aligned}$$

In the latter case we use easily checked identity

$$\bar{a}^{-1}(i-1) = z^{k-1} a^{[-k]}(ki).$$

PROPOSITION 5. *The set of transformation (37) with superposition operation is isomorphic to multiplicative semi-group \mathbb{Z}^* .*

PROOF. By virtue (38), we have

$$\begin{aligned} g_r \circ g_k(a(i)) &= z^{1-r} \bar{a}^{[r]}(ri) = \\ &= z^{1-r} z^{r(1-k)} a^{[rk]}(rki) = z^{1-rk} a^{[rk]}(rki) = g_{rk}(a(i)). \end{aligned}$$

PROPOSITION 6. *(37) is symmetry transformation for DKP chain.*

PROOF. By virtue of Proposition 1, we have

$$\partial_p \bar{a}(i) = z^{1-k} \partial_p a^{[k]}(ki) =$$

$$= z^{1-k} a^{[k]}(ki) (H^{(p)}(ki+k) - H^{(p)}(ki)) = \bar{a}(i) (\bar{H}^{(p)}(i+1) - \bar{H}^{(p)}(i)).$$

In addition,

$$\partial_p \bar{h}(i) = \partial_p h(ki) = \partial H^{(p)}(ki) = \partial \bar{H}^{(p)}(i).$$

PROPOSITION 7. *Let $\{h(i), a(i)\} \in \mathcal{S}_l^{kn}$, then $\{g_k(h(i)), g_k(a(i))\} \in \mathcal{S}_l^n$*

PROOF. If $z^{l-kn+1} a^{[kn]}(i) \in \mathcal{H}_+(i)$, then by virtue (38) we have

$$z^{l-n+1} \bar{a}(i) = z^{l-n+1} z^{n(1-k)} a^{[kn]}(ki) = z^{l-kn+1} a^{[kn]}(ki) \in \mathcal{H}_+(ki) = \bar{\mathcal{H}}_+(i).$$

So, we can write $g_k(\mathcal{S}_l^{kn}) \subset \mathcal{S}_l^n$.

It is natural that the transformation (37) affects corresponding transformation of the functions $q_k^{(n,r)}(i)$. To find out the rule of how they transform we use (16). By virtue of (38), we have

$$\begin{aligned} z^r &= \bar{a}^{[r]}(i) + \sum_{s \geq 1} \bar{q}_s^{(n,r)}(i) z^{s(n-1)} \bar{a}^{[r-sn]}(i) = \\ &= z^{r(1-k)} a^{[rk]}(ki) + \sum_{s \geq 1} \bar{q}_s^{(n,r)}(i) z^{s(n-1)} z^{(r-sn)(1-k)} a^{[(r-sn)k]}(ki). \end{aligned}$$

Multiplying both sides of this relation by $z^{r(k-1)}$, we obtain

$$z^{rk} = a^{[rk]}(ki) + \sum_{s \geq 1} \bar{q}_s^{(n,r)}(i) z^{s(kn-1)} a^{[kr-skn]}(ki).$$

It follows from this following identification

$$\bar{q}_s^{(n,r)}(i) = q_s^{(kn,kr)}(ki). \quad (39)$$

Observe, that if one wants to get transformation in the form of the mapping $\{q_s^{(kn,r)}(i)\} \rightarrow \{\bar{q}_s^{(n,r)}(i)\}$, there is a need to make use the formula (24). As an example, take $n = 1, r = 1, k = 2$. Let $\bar{q}_s(i) \equiv \bar{q}_s^{(1,1)}(i)$ and $q_s(i) \equiv q_s^{(2,1)}(i)$, then

$$\bar{q}_1(i) = q_1(2i) + q_1(2i+1), \quad \bar{q}_2(i) = q_2(2i) + q_1(2i-1)q_1(2i) + q_1(2i+1),$$

$$\bar{q}_3(i) = q_3(2i) + q_1(2i)q_2(2i-1) + q_1(2i-3)q_2(2i) + q_3(2i+1)$$

and so on. These relation serve to map the solutions of gap KP hierarchy, for $n = 2$, to solutions of discrete KP hierarchy. In particular, if $q_s(i) \equiv 0$ $s \geq 2$, one gets well-known Miura transformation between Volterra and Toda lattices

$$\bar{q}_1(i) = q_1(2i) + q_1(2i+1), \quad \bar{q}_2(i) = q_1(2i-1)q_1(2i). \quad (40)$$

Observe, that g_k only for $k = -1$ and $k = 1$ is invertible transformation. It is natural, for $k \neq \pm 1$, (37) (or (39)) to call lattice Miura transformation.

PROPOSITION 8. *Let the number $kln - r$ do not multiply by k ; the DKP chain solution is in $\mathcal{S}_0^{kn} \cap \mathcal{S}_{l-1}^{kln-r}$; the submanifolds \mathcal{S}_0^{kn} and $\mathcal{S}_{l-1}^{kln-r}$ are origins of inclusions chains for given solution, then $g_k(\mathcal{S}_0^{kn} \cap \mathcal{S}_{l-1}^{kln-r}) \subset \mathcal{S}_0^n \cap \mathcal{S}_{kl-1}^{kln-r}$. Moreover \mathcal{S}_0^n and $\mathcal{S}_{kl-1}^{kln-r}$ are origins of inclusions chains for transformed solution.*

PROOF. By condition, we have two chains of submanifolds

$$\begin{aligned}\mathcal{S}_0^{kn} &\subset \mathcal{S}_1^{2kn} \subset \dots \\ \mathcal{S}_{l-1}^{lkn-r} &\subset \mathcal{S}_{2l-1}^{2(lkn-r)} \subset \dots \subset \mathcal{S}_{kl-1}^{k(lkn-r)} \subset \dots\end{aligned}$$

consisting given solution of DKP chain. By virtue of Proposition 7, transformed solution is in submanifolds

$$\begin{aligned}\mathcal{S}_0^n &\subset \mathcal{S}_1^{2n} \subset \dots \\ \mathcal{S}_{kl-1}^{lkn-r} &\subset \mathcal{S}_{2kl-1}^{2(lkn-r)} \subset \dots\end{aligned}$$

So, the proposition is proved.

By virtue of this proposition one can write

$$g_k(\mathcal{S}_{kn,r,l}) \subset \mathcal{S}_{n,r,kl}.$$

Together with (39) Proposition 8 is a strong basis for constructing of Miura transformations between lattices with finite number of fields. Simplest example is given by (40). We learn, for example, from Proposition 8, that Bogoyavlenskii lattices (28) are connected by Miura transformations with generalized Toda ones (29). In this case one can write $g_n(\mathcal{S}_{n,1,1}) \subset \mathcal{S}_{1,1,n}$, $n \geq 2$.

Some examples of lattice Miura transformations which connect one-component lattices to multi-component ones can be found in [16], [17].

5 Self-similar solutions

5.1. Invariant solutions.

It is evident, that linear systems (20), (21) and its consistency relations (23) and (24) are invariant under group of dilatations

$$q_k^{(n,r)}(i) \rightarrow \epsilon^k q_k^{(n,r)}(i), \quad t_l \rightarrow \epsilon^{-l} t_l, \quad z \rightarrow \epsilon z, \quad \Psi_i \rightarrow \epsilon^i \Psi_i.$$

In what follows we consider dependencies only on finite number of evolution parameters t_1, \dots, t_p . Invariants of this group are

$$T_l = \frac{t_l}{(pt_p)^{l/p}}, \quad l = 1, \dots, p-1, \quad \xi = (pt_p)^{1/p} z, \quad \psi_i = z^i \Psi_i.$$

From this we get the ansatzes for self-similar solutions:

$$q_k^{(n,r)}(i) = \frac{1}{(pt_p)^{k/p}} x_k^{(n,r)}(i), \tag{41}$$

$$\Psi_i = z^i \psi_i(\xi; T_1, \dots, T_{p-1}).$$

Here $x_k^{(n,r)}(i)$'s are unknown functions of T_1, \dots, T_{p-1} . Direct substitution of (41) into (23) gives

$$\partial_{T_l} x_k^{(n,r)}(i) = X_{k,l}^{(n,r)}(i), \quad l = 1, \dots, p-1 \quad (42)$$

and

$$\begin{aligned} kx_k^{(n,r)}(i) + T_1 X_{k,1}^{(n,r)}(i) + 2T_2 X_{k,2}^{(n,r)}(i) + \dots + (p-1)T_{p-1} X_{k,p-1}^{(n,r)}(i) + \\ + X_{k,p}^{(n,r)}(i) = 0, \quad l = 1, \dots, p-1. \end{aligned} \quad (43)$$

Here $X_{k,l}^{(n,r)}(i)$'s are RHS's of (23) where $q_k^{(n,r)}(i)$'s are replaced by $x_k^{(n,r)}(i)$'s. Moreover it follows from (24) that

$$\begin{aligned} x_k^{(n,r_1+r_2)}(i) &= x_k^{(n,r_1)}(i) + \sum_{s=1}^{k-1} x_s^{(n,r_1)}(i) x_{k-s}^{(n,r_2)}(i + r_1 - sn) + q_k^{(n,r_2)}(i + r_1) = \\ &= x_k^{(n,r_2)}(i) + \sum_{s=1}^{k-1} x_s^{(n,r_2)}(i) x_{k-s}^{(n,r_1)}(i + r_2 - sn) + x_k^{(n,r_1)}(i + r_2). \end{aligned} \quad (44)$$

Corresponding auxiliary linear equations are transformed to the following form:

$$\begin{aligned} \partial_{T_l} \psi &= (X_{(n)}^{ln})_+ \psi, \quad l = 1, \dots, p-1, \\ \xi \psi_\xi &= \left\{ T_1 (X_{(n)}^n)_+ + 2T_2 (X_{(n)}^{2n})_+ + \dots + (p-1) (X_{(n)}^{(p-1)n})_+ + (X_{(n)}^{pn})_+ \right\} \psi, \\ X_{(n)}^r \psi &= \xi \psi, \end{aligned}$$

where

$$X_{(n)}^r \equiv \xi \Lambda^r + \sum_{k \geq 1} \xi^{1-k} x_k^{(n,r)} \Lambda^{r-kn}, \quad r \in \mathbb{Z}.$$

5.2. Examples.

Let us consider, as a simple example, the case corresponding Volterra lattice hierarchy, that is $n = 2, r = 1, l = 1$. Take $p = 2$. The equations (42) and (43) are written down as follows:

$$x'_i = x_i(x_{i+1} - x_{i-1}), \quad ' \equiv \partial / \partial T_1, \quad (45)$$

$$x_i + T_1 x_i \left\{ x_1^{(2,2)}(i) - x_1^{(2,2)}(i-1) \right\} + x_i \left\{ x_2^{(2,4)}(i) - x_2^{(2,4)}(i-1) \right\} = 0. \quad (46)$$

Here we denote $x_i = x_1^{(2,1)}(i)$. Using (44) one calculates

$$x_1^{(2,2)}(i) = x_i + x_{i+1}, \quad x_2^{(2,4)}(i) = x_i(x_{i-1} + x_i + x_{i+1}) + x_{i+1}(x_i + x_{i+1} + x_{i+2}).$$

Taking into account these relations, the equation (46) turns into

$$\begin{aligned} T_1 x_i + x_i(x_{i-1} + x_i + x_{i+1}) &= \alpha_i, \\ 1 + \alpha_{i+1} - \alpha_{i-1} &= 0. \end{aligned} \quad (47)$$

Let us prove that by virtue of (45) α_i 's constants. Indeed, we have

$$\alpha'_i = x_i + T_1 x_i(x_{i+1} - x_{i-1}) +$$

$$\begin{aligned}
& +x_i(x_{i+1} - x_{i-1})(x_{i-1} + x_i + x_{i+1}) + x_i(x_{i+1}x_{i+2} - x_{i-1}x_{i-2}) = \\
& = x_i(1 + \alpha_{i+1} - \alpha_{i-1}) = 0.
\end{aligned}$$

So, one can rewrite (47) as

$$x_{i-1} + x_i + x_{i+1} = -T_1 + \frac{\alpha_i}{x_i}. \quad (48)$$

Here α_i 's are constants forced by the condition $\alpha_{i+2} = \alpha_i - 1$. One can write the solution of this equation as $\alpha_i = \alpha - \frac{1}{2}i + \beta(-1)^i$ where α and β are some constants. Provided these conditions, (48) is dPI [27].

Observe that evolution equation (45) with (48) turns into

$$x'_i = 2x_i x_{i+1} + x_i^2 + T_1 x_i - \alpha_i.$$

It can be easily checked that together with (48) this lattice is equivalent to the pair of ordinary first-order differential equations

$$\begin{aligned}
w'_1 &= 2w_1 w_2 + w_1^2 + T_1 w_1 + a, \\
w'_2 &= -2w_1 w_2 - w_2^2 - T_1 w_2 - b
\end{aligned} \quad (49)$$

with discrete symmetry transformation

$$\bar{w}_1 = w_2, \quad \bar{w}_2 = -w_1 - w_2 - T_1 - \frac{b}{w_2}, \quad \bar{a} = b, \quad \bar{b} = a + 1,$$

where $w_1 \equiv x_i$, $w_2 = x_{i+1}$, $a \equiv -\alpha_i$, $b \equiv -\alpha_{i+1}$ for some fixed (but arbitrary) value $i = i_0$. In turn the system (49) is equivalent to second-order equation

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 2T_1 w^2 + \left(\frac{T_1^2}{2} + a - 2b + 1 \right) w - \frac{a^2}{2w}, \quad w \equiv w_1$$

with corresponding symmetry transformation

$$\bar{w} = \frac{w' - w^2 - T_1 w - a}{2w}, \quad \bar{a} = b, \quad \bar{b} = a + 1.$$

In fact this is PIV with Bäcklund transformation [28], [29]. By dilatations $T_1 \rightarrow \sqrt{2}T_1$, $w \rightarrow w/\sqrt{2}$ it can be turned to following canonical form:

$$\begin{aligned}
w'' &= \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4T_1 w^2 + 2(T_1^2 + a - 2b + 1)w - \frac{2a^2}{w} \\
\bar{w} &= \frac{w' - w^2 - 2T_1 w - 2a}{2w}, \quad \bar{a} = b, \quad \bar{b} = a + 1.
\end{aligned} \quad (50)$$

In a result, we obtain the well-known relationship between dPI (48) and PIV (50) [33].

REMARK 7. The equations (49) can be interpreted as self-similar reduction of Levi system [34]

$$\begin{aligned}
v_{1t_2} &= (-v'_1 + v_1^2 + 2v_1 v_2)', \\
v_{2t_2} &= (v'_2 + v_2^2 + 2v_1 v_2)'
\end{aligned}$$

with the help of the ansatze

$$v_k = \frac{1}{(2t_2)^{1/2}} w_k(T_1), \quad k = 1, 2.$$

This fact is suitable for results of the work [17], where we established the correspondence between some class of one-component lattices and hierarchies of evolution equations which can be interpreted as modified version of Krichever's rational reductions of KP hierarchy. In particular, Levi system hierarchy corresponds to Volterra lattice.

Let us consider more general case corresponding to Bogoyavlenskii lattices (28). The analogues of the equations (45) and (46) in this case are Volterra lattice.

$$x'_i = x_i \left(\sum_{s=1}^{n-1} x_{i+s} - \sum_{s=1}^{n-1} x_{i-s} \right), \quad (51)$$

$$x_i + T_1 x_i \left\{ x_1^{(n,n)}(i) - x_1^{(n,n)}(i+1-n) \right\} + x_i \left\{ x_2^{(n,2n)}(i) - x_2^{(n,2n)}(i+1-n) \right\} = 0. \quad (52)$$

Moreover we take into account that

$$x_1^{(n,n)}(i) = \sum_{s=1}^n x_{i+s-1}, \quad x_2^{(n,2n)}(i) = \sum_{s=1}^n x_{i+s-1} \left(\sum_{s_1=1}^{2n-1} x_{i+s_1+s-n-1} \right).$$

Then the equation (52) is rewritten in the form

$$\begin{aligned} T_1 x_i + x_i (x_{i+1-n} + \dots + x_{i+n-1}) &= \alpha_i, \\ 1 + \sum_{s=1}^{n-1} \alpha_{i+s} - \sum_{s=1}^{n-1} \alpha_{i-s} &= 0. \end{aligned} \quad (53)$$

It can be proved, that by virtue of (51),

$$\alpha'_i = x_i \left(1 + \sum_{s=1}^{n-1} \alpha_{i+s} - \sum_{s=1}^{n-1} \alpha_{i-s} \right) = 0.$$

So, one concludes that in this case self-similar ansatze leads to equation

$$x_{i+1-n} + \dots + x_{i+n-1} = -T_1 + \frac{\alpha_i}{x_i}, \quad (54)$$

when the constants α_i is connected with each other by (53).

Standard analysis of singularity confinement shows that this property for (54) is valid provided that

$$\alpha_{i+n} - \alpha_i = \alpha_{i+2n-1} - \alpha_{i+n-1}. \quad (55)$$

This equation do not contradict to (53), but is more general. Let us show it. It follows from (53) that

$$-1 = \sum_{s=1}^n \alpha_{i+s+n-1} - \sum_{s=1}^n \alpha_{i+s} = \sum_{s=1}^n \alpha_{i+s+n-2} - \sum_{s=1}^n \alpha_{i+s-1}. \quad (56)$$

Second equality in (56) can be rewritten in the form

$$\sum_{s=2}^{n-1} \alpha_{i+s+n-1} + \alpha_{i+2n-1} - \sum_{s=1}^{n-2} \alpha_{i+s} - \alpha_{i+n-1} = \alpha_{i+n} + \sum_{s=3}^n \alpha_{i+s+n-2} - \alpha_i - \sum_{s=2}^{n-1} \alpha_{i+s-1}.$$

From the latter one obtains (55).

As is known, Bogoyavlenskii lattice (28), for any n , can be interpreted as integrable discretization of Korteweg-de Vries equation. Similarly, autonomous version of (54), for any n , is integrable discretization of PI: $w'' = 6w^2 + t$. Let us prove this. Let $\alpha_i = \alpha$. One divides real axis into segments of equal length ε . So, it can be written $t = i\varepsilon$. Values of the function w , respectively, is taken for all such values of the variable t . Then one can denote $w(t) = w_i$. Let

$$x_i = 1 + \varepsilon^2 w_i, \quad \alpha = 1 - 2n - \varepsilon^4 t, \quad T_1 = -(2n - 1). \quad (57)$$

Substituting (57) in the equation (54), taking into account the relations of the form

$$x_{i+1} = 1 + \varepsilon^2 w_{i+1} = 1 + \varepsilon^2 \left\{ w + \varepsilon w' + \frac{\varepsilon^2}{2} w'' + \dots \right\}$$

and turning then ε to zero we obtain, in continuous limit the equation

$$\sum_{s=1}^{n-1} (n-s)^2 \cdot w'' = -t - (2n-1)w^2$$

which by dilatations can be deduced to canonical form of PI.

To conclude the paper, we exhibit two examples of integrable mappings and suitable Miura transformations.

EXAMPLE 8. Two-component mapping

$$\begin{aligned} \beta_i &= T_1 y_1(i) + y_1^2(i) + y_2(i) + y_2(i+1), \\ y_1(i) + y_2(i+1)(y_1(i) + y_1(i+1) + T_1) - y_2(i)(y_1(i) + y_1(i-1) + T_1) &= 0, \\ 2 + \beta_i - \beta_{i-1} &= 0. \end{aligned} \quad (58)$$

is suitable for Toda lattice hierarchy. The connection with dPI (47) is given by Miura transformation

$$y_1(i) = x_{2i} + x_{2i+1}, \quad y_2(i) = x_{2i-1} x_{2i}, \quad \beta_i = \alpha_{2i} + \alpha_{2i+1}. \quad (59)$$

For example, the substitution of (59) into (58) gives

$$x_{2i}(1 + \alpha_{2i+1} - \alpha_{2i-1}) + x_{2i+1}(1 + \alpha_{2i+2} - \alpha_{2i}) = 0.$$

EXAMPLE 9. Discrete equations

$$\beta_i = T_1 y_1(i) + y_1^2(i) + y_2(i) + y_2(i+1),$$

$$\begin{aligned}
y_1(i)+y_2(i+1)(y_1(i)+y_1(i+1)+T_1)-y_2(i)(y_1(i)+y_1(i-1)+T_1)+y_3(i+2)-y_3(i) &= 0, \quad (60) \\
y_2(i)(2+\beta_i-\beta_{i-1})+y_3(i+1)(y_1(i)+y_1(i+1)+T_1)- \\
-y_3(i)(y_1(i-1)+y_1(i-2)+T_1) &= 0, \\
3+\beta_i-\beta_{i-2} &= 0.
\end{aligned}$$

correspond to three-component generalized Toda lattice (29) hierarchy. The system (60) is connected by Miura transformation

$$y_1(i) = x_{3i} + x_{3i+1} + x_{3i+2}, \quad y_2(i) = x_{3i-2}x_{3i} + x_{3i-1}x_{3i} + x_{3i-1}x_{3i+1}, \quad y_3(i) = x_{3i-4}x_{3i-2}x_{3i},$$

$$\beta_i = \alpha_{3i} + \alpha_{3i+1} + \alpha_{3i+2}$$

with (54), for $n = 3$.

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